Persistent currents for Coulomb interacting electrons on 2d disordered lattices

Sign and interaction dependence in the Wigner crystal regime

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Abstract. A Wigner crystal structure of the electronic ground state is induced by strong Coulomb interactions at low temperature in clean or disordered two-dimensional (2d) samples. For fermions on a mesoscopic disordered 2d lattice, being closed to a torus, we study the persistent current in the regime of strong interaction at zero temperature. We perform a perturbation expansion starting from the Wigner crystal limit which yields power laws for the dependence of the persistent current on the interaction strength. The sign of the persistent current in the strong interaction limit is independent of the disorder realization and strength. It depends only on the electro-statically determined configuration of the particles in the Wigner crystal.

1 Introduction

The interplay of disorder and interaction in mesoscopic samples has attracted considerable interest in recent years [1]. A prominent experimental finding in this field is the insulator-metal transition in 2d systems of high mobility [2], occurring when the carrier density is increased. This transition can neither be explained by disorder effects alone nor by interaction effects in clean samples and is the subject of still increasing experimental and theoretical activities (for a review see [3] and references therein).

One possible mechanism invokes both, interaction and disorder and associates the insulator–metal transition with the melting of a pinned Wigner crystal when the carrier density increases and therewith r_s (the interaction energy in units of the kinetic energy) decreases, allowing for metallic behavior at intermediate r_s. Only at a higher density, when r_s is small, the interaction becomes negligible and the now dominating disorder-induced Anderson localization leads back to insulating behavior. Such a scenario is supported by experiments [4] and numerical investigations of few interacting particles in disordered lattice models [5–7].

Another important experimental result in mesoscopic physics whose explanation necessitates to invoke disorder and interactions simultaneously is the value of the persistent current in diffusive rings [8]. These persistent currents are much larger than the theoretical prediction for non-interacting electrons in disordered rings [9]. While the electron-electron interaction seems to play an essential role, the disorder in the sample is also important: Interactions cannot affect the persistent current in clean rotationally invariant 1d rings [10,11], and the non-interacting result is consistent with the experimental one for a clean semiconductor ring in the ballistic regime [12].

This has generated a large theoretical activity, dealing with the combined effect of interactions and disorder on the enhancement of persistent currents in mesoscopic rings (for an overview see *e.g.* [1,13,14] and references therein). Even though different theoretical approaches suggest an increase of the persistent current in disordered samples due to repulsive Coulomb interactions, a quantitative understanding of the experiments is still lacking.

Within *continuous* models of fermions in 1d with disorder, repulsive interactions are found to enhance the persistent currents, without [11] and with [15] spin. On moderately disordered *lattices*, repulsive interactions are however found to decrease the persistent current for spinless fermions in 1d [16–18]. It was concluded from analytical considerations [19], using renormalization group arguments, that 1d lattice models exhibit an interaction-induced enhancement of the persistent current only when the spin degree of freedom is taken into account. Nevertheless, a numerical study of small systems [20] revealed that repulsive interactions can slightly enhance

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the averaged persistent current even for spinless fermions on a 1d lattice, provided the disorder is very strong.

A recent numerical investigation treating the persistent current for strongly disordered *individual* 1d chains [21,22] leads to the conclusion that, while Anderson localization is dominating the non-interacting case, the persistent current can be *strongly enhanced* by repulsive local interactions at sample dependent intermediate values of the interaction strength. This delocalization is accompanied by a reorganization of the ground state structure. At half filling, strong interactions induce a regular charge density in the Mott-insulator regime and decrease the persistent current.

This suppression of the persistent current in the limit of strong repulsive interactions is not limited to 1d models of spinless fermions at half-filling with local interactions. It occurs with long-range interactions [16] at arbitrary filling and also in 2d lattices [5], when strong interaction (large r_s) leads to the formation of a Wigner crystal pinned by the disorder in the ground state.

In the case of the Hubbard model at half filling [23] and for spinless fermions in 1d chains without [24] and with disorder [25], this suppression of the persistent current at strong interaction has been understood quantitatively from a perturbation theory starting at the Mott insulator limit. In contrast, numerical Hartree-Fock approaches which allow to treat larger systems than the exact diagonalization used in references [5,16] and which have been used for weakly interacting particles in 1d and 2d [26], are, at least in 1d, not able to quantitatively describe the persistent current in the limit of strong interaction [27].

The sign of the persistent current in the case of spinless particles in strictly 1d rings is independent on the disorder realization and the interaction strength. According to a well-known theorem by Leggett [28], the sign is given solely by the parity of the particle number N (paramagnetic for N even and diamagnetic for N odd). This general rule is confirmed by an explicit calculation for a Luttinger liquid ring without disorder [29].

On the other hand, the sign of the persistent current for particles with spin in disordered 1d rings or spinless fermions in disordered 2d systems differs from sample to sample and no general sign rule exists. Only in some special cases, like non-interacting particles with spin in clean 1d rings [30], and for some interacting situations using the Hubbard model [31,32], the sign of the persistent current has been determined. The situation is less clear in the presence of long-range interactions and disorder we address in this work. For electrons with spin in 1d rings having a particular disorder consisting of only one barrier, a tendency towards diamagnetic responses, independent of the particle number, was found at strong repulsive interaction [33].

For spinless fermions in strongly disordered 2d lattice models, it has been noticed in numerical studies that the sign of the persistent current becomes realization-independent in the limit of strong interaction [5,6,34,35]. Detailed studies of the local current show that the suppression of its transverse component by the interactions is much stronger than the decrease of the longitudinal

current. On such a lattice, closed to a torus, the structure of the ground state at strong interaction is a Wigner crystal pinned by the disorder [5,6,35]. While the system exhibits Anderson localization at weak interaction, the regime of intermediate interaction shows indications of a new type of correlated metal [36].

For spinless fermions in 2d lattice models *without disorder*, the amplitude of the persistent current has recently been studied analytically and numerically at strong interaction [37]. When r_s is large, the hopping matrix elements between neighboring lattice sites being much smaller than the interaction strength, the behavior can be understood from a perturbation theory in terms of the hopping matrix elements.

In this paper we report a study of the persistent current in *disordered* 2d lattice models at *very strong* Coulomb interaction, using a perturbation theory expansion around the pinned Wigner crystal. While the understanding of this regime cannot directly explain the high persistent currents observed experimentally in diffusive metal rings, it may be relevant for the insulating side at low carrier density of the insulator-metal transition in 2d. We show how the sign of the persistent current at strong interaction follows systematically from the structure of the Wigner crystal and find simple rules for this sign. For the absolute values, power laws similar to the ones found in [37] are obtained also in the disordered case. We shall show how the presence of disorder and spin can influence the prefactors and the exponents of these power laws.

In the following section we introduce the model for interacting fermions on a disordered lattice and the quantities used to characterize its properties. The perturbation theory is developed in Section 3 and applied in Section 4 to the persistent current in longitudinal and transverse direction before we conclude the paper.

2 Model

2.1 Hamiltonian

We consider N fermions on a disordered square lattice with Coulomb interaction. The corresponding Hamiltonian reads

$$H = H_K + H_D + H_I. \tag{1}$$

The kinetic energy term is

$$H_K = -t \sum_{\langle i,i' \rangle} \sum_\sigma c^+_{i,\sigma} c_{i',\sigma} \tag{2}$$

with the hopping matrix element $t = 1$ setting the energy scale. We concentrate on rectangular 2d lattice structures with $L_x \times L_y$ sites i. The fermionic on-site operators $c_{i,\sigma} \equiv c_{(x_i,y_i),\sigma}$ destroy a particle with spin σ located at $\boldsymbol{r}_i = (x_i, y_i)$, where the position coordinates $x_i \in \{1, 2, \ldots, L_x\}$ and $y_i \in \{1, 2, \ldots, L_y\}$ are measured in units of the lattice spacing a.

The disordered potential contribution is

$$H_D = W \sum_{i,\sigma} v_i \hat{n}_{i,\sigma}, \tag{3}$$

where W is the disorder strength, with the independent random variables v_i, drawn from a box distribution within the interval $[-1/2; +1/2]$. The occupation number operators are as usual given by $\hat{n}_{i,\sigma} = c_{i,\sigma}^+ c_{i,\sigma}$. The Coulomb interaction is described by the term

$$H_{\mathrm{I}} = \frac{U}{2} \sum_{\substack{i,i' \\ i \neq i'}} \sum_{\sigma,\sigma'} \frac{\hat{n}_{i,\sigma} \hat{n}_{i',\sigma'}}{|\boldsymbol{r}_i - \boldsymbol{r}_{i'}|} + U \sum_i \frac{\hat{n}_{i,\uparrow} \hat{n}_{i,\downarrow}}{d}, \qquad (4)$$

the interaction strength being parametrized by U. The sum in H_{K} runs over all pairs of sites which are next neighbors on the lattice $\langle i, i' \rangle$, while the interaction term is composed of a sum over all pairs of different sites. With these definitions, one finds $r_{\mathrm{s}} = \frac{U}{2t\sqrt{\pi\nu}}$, with the electronic density or filling factor $\nu = N/L_x L_y = 1/b^2$, b being the average distance between particles. An additional term takes into account double occupancy of a site by two particles of different spin, $d \, (< a)$ being a measure for the size of the on-site orbitals.

In order to study the persistent current, we close the 2d lattice first to a cylinder by imposing generalized periodic boundary conditions

$$c_{(L_x,y);\sigma} = \exp(\mathrm{i}\phi_x) c_{(0,y);\sigma} \qquad (5)$$

in x-direction. For $\phi_x = 0$, this is equivalent to usual periodic boundary conditions. Finite ϕ_x accounts for a magnetic flux $\Phi = \phi_x \Phi_0/2\pi$ threading the ring, Φ_0 being the flux quantum. We choose the units such that $\Phi_0/2\pi = 1$. In order to reduce finite size effects, we use periodic boundary conditions in y-direction, and thus obtain a torus topology with the fluxes $(\phi_x, \phi_y) = (\phi_x, 0)$. In Section 4.3 we will use the dependence on the transverse flux ϕ_y to study also the transverse current.

2.2 Persistent current

The magnetic flux threading the ring can drive a persistent current through the system. At zero temperature, it is given by

$$I(\phi_x) = -\left. \frac{\partial E_0}{\partial \phi} \right|_{\phi = \phi_x}, \qquad (6)$$

where E_0 is the many-particle ground state energy. Thus, the persistent current at $T = 0$ is a measure of the dependence of the ground state energy on the magnetic flux. Since the latter can be expressed in the form of a boundary condition, it is at the same time a measure of the ground state sensitivity to the boundary conditions and can be related to the conductance of the sample [38].

3 Theoretical approach

3.1 Wigner crystal at strong interaction

In the non-interacting limit, disorder leads to Anderson localization of the one-particle states and the problem can be treated by a perturbative expansion around the on-site localized states in terms of the hopping matrix elements t [39]. Hopping to distant sites costs disorder energy of the order of W such that a series expansion in t/W results.

In the many-body case, strong repulsive interaction U leads to Wigner crystallization of the ground state with on-site localized charges in the electro-statically most favorable position. One can use a similar perturbative formalism in terms of the hopping t, but now, the essential cost in energy caused by displacing one of the many particles is given by the increase of the interaction energy such that one obtains a systematic expansion in terms of t/U.

We decompose the Hamiltonian (1) as

$$H = H_0 + H_{\mathrm{K}} \qquad (7)$$

with an unperturbed part containing disorder and interaction

$$H_0 = H_{\mathrm{D}} + H_{\mathrm{I}}, \qquad (8)$$

and the perturbation given by the hopping terms of H_{K}.

H_0 is composed of terms containing only occupation number operators in the one-particle on-site basis. Therefore, its N-particle eigenstates $|\psi_\alpha\rangle$ are Slater determinants built from N different one-particle functions and are completely characterized by the occupation numbers $n_{i,\sigma}(\alpha) \in \{0, 1\}$ of the one-particle states on site i with spin σ, fulfilling the condition $N = \sum_{i,\sigma} n_{i,\sigma}(\alpha)$. Therewith, the many-body eigenstates of H_0 can be written in the form

$$|\psi_\alpha\rangle = \left(\prod_{i,\sigma} (c_{i,\sigma}^+)^{n_{i,\sigma}(\alpha)} \right) |0\rangle. \qquad (9)$$

($|0\rangle$ is the vacuum state), and the corresponding eigenenergies are given by $E_\alpha = E_\alpha^{\mathrm{D}} + E_\alpha^{\mathrm{I}}$ with

$$E_\alpha^{\mathrm{D}} = W \sum_{i,\sigma} v_i n_{i,\sigma}(\alpha) \qquad \text{and} \qquad (10)$$

$$E_\alpha^{\mathrm{I}} = \frac{U}{2} \sum_{\substack{i,i' \\ i \neq i'}} \sum_{\sigma,\sigma'} \frac{n_{i,\sigma}(\alpha) n_{i',\sigma'}(\alpha)}{|\boldsymbol{r}_i - \boldsymbol{r}_{i'}|} + U \sum_i \frac{n_{i,\uparrow} n_{i,\downarrow}}{d}. \qquad (11)$$

The ground state $|\psi_0\rangle$ of this Coulomb glass problem is given by purely classical considerations, minimizing disorder and interaction energy. Its charge configuration depends in general on the specific disorder realization of the sample. At strong enough interaction, when the disorder effects are dominated by the interaction, the structure of $|\psi_0\rangle$ is the Wigner crystal of minimal interaction energy, independent of the disorder realization. The rigid array of charges can be translated as a whole through the system without changing the interaction energy. Nevertheless, the contribution of the disordered potential to the energy depends on the realization and pins the Wigner crystal in a realization dependent position. It is important to realize that the structure of the Wigner crystal is entirely given by the lattice geometry and the Coulomb interaction of the N particles.

In contrast to H_0, the perturbing part H_{K} of the Hamiltonian depends on the magnetic flux through

the ring since the latter appears in the boundary condition in x-direction and therefore influences some hopping matrix elements. Writing all the hopping terms explicitly, one obtains

$$
H_{\mathrm{K}}(\phi_x, \phi_y) = -t \sum_\sigma \left(\sum_{x=1}^{L_x} \sum_{y=2}^{L_y} c^+_{(x,y);\sigma} c_{(x,y-1);\sigma} \right.
$$

$$
+ \sum_{x=2}^{L_x} \sum_{y=1}^{L_y} c^+_{(x,y);\sigma} c_{(x-1,y);\sigma} + \mathrm{e}^{-\mathrm{i}\phi_y} \sum_{x=1}^{L_x} c^+_{(x,1);\sigma} c_{(x,L_y);\sigma}
$$

$$
\left. + \mathrm{e}^{-\mathrm{i}\phi_x} \sum_{y=1}^{L_y} c^+_{(1,y);\sigma} c_{(L_x,y);\sigma} + \mathrm{H.C.} \right) . \tag{12}
$$

3.2 Perturbation expansion for spinless fermions

For electrons with spin, the unperturbed ground state is 2^N-fold degenerate since all the spin configurations yield the same energy when hopping is suppressed. Further degeneracies appear in the case of clean systems when $W = 0$ allows for translational symmetry.

For simplicity we first treat the case of completely spin polarized systems (all spins up, equivalent to spinless fermions) with disorder where the ground state of H_0 is not degenerate and the expansion in H_{K} is straightforward.

Using standard perturbation theory, the correction to the ground state energy in nth order is given by

$$
E_0^{(n)} = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}} \tag{13}
$$

$$
\frac{\langle \psi_0 | H_{\mathrm{K}} | \psi_{\alpha_1} \rangle \langle \psi_{\alpha_1} | H_{\mathrm{K}} | \psi_{\alpha_2} \rangle \ldots \langle \psi_{\alpha_{n-1}} | H_{\mathrm{K}} | \psi_0 \rangle}{(E_0 - E_{\alpha_1})(E_0 - E_{\alpha_2}) \ldots (E_0 - E_{\alpha_{n-1}})},
$$

with the sums running over all the eigenstates of H_0 except the ground state itself.

The numerator of the contributions to the sum of equation (13) contains matrix elements $\langle \psi_{\alpha_k} | H_{\mathrm{K}} | \psi_{\alpha_{k+1}} \rangle$ of the perturbing Hamiltonian. Since H_{K} consists only of one-particle hopping terms, non-zero matrix elements can arise only if the two states $|\psi_{\alpha_k}\rangle$ and $|\psi_{\alpha_{k+1}}\rangle$ differ by nothing else than a single hop of one of the particles. From (13) one sees that a finite contribution to the sum over different sequences of intermediate states α is obtained only if the n one-particle hops are such that the final configuration has an overlap with the initial one, corresponding to the ground state. The n sums over intermediate states α_k in equation (13) can then be rewritten as a sum over all the sequences $S = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ which give a nonzero contribution. Denoting the numerator of the terms by $\mathrm{Num}(S)$ and the denominator by $\mathrm{Den}(S)$, equation (13) takes the form

$$
E_0^{(n)} = \sum_S \frac{\mathrm{Num}(S)}{\mathrm{Den}(S)}. \tag{14}
$$

We will now evaluate the numerator and the denominator separately. The numerator $\mathrm{Num}(S)$ can be calculated directly from the n hopping matrix elements, thereby taking into account the flux dependent phase for hops crossing the boundary. Since we consider fermions, the corresponding operators anti-commute and their order in the products (9) defining the basis states $|\psi_\alpha\rangle$ is crucial for the sign. When the starting point (the ground state configuration) is re-established after n consecutive hopping processes, the order of the operators can be modified. Then, the sign of the permutation P_S of the operators, caused by the sequence of one-particle hops S, must be incorporated in the result. Altogether, one obtains

$$
\mathrm{Num}(S) = \mathrm{sign}(P_S) \, (-t)^n \exp\left[-\mathrm{i}\phi_x (h_{\mathrm{f}} - h_{\mathrm{b}})\right] . \tag{15}
$$

h_{f} and h_{b} denote the number of hoppings across the boundary between sites (L_x, y) and $(1, y)$ in forward and backward direction, respectively. Therefore, only the corrections to the ground state energy due to sequences with $h_{\mathrm{f}} - h_{\mathrm{b}} \neq 0$ are flux dependent.

Moving particles creates defects in the Wigner crystal. This increases the interaction energy (see Eq. (11)) of the ground state E_0^{I} by the amount $U\epsilon_\alpha$, where

$$
\epsilon_\alpha = \frac{1}{U} \left(E_\alpha^{\mathrm{I}} - E_0^{\mathrm{I}}\right) \tag{16}
$$

is non-negative and independent of U. It accounts for the difference of the inter-particle distances between the configurations of the state $|\psi_\alpha\rangle$ and the ground state $|\psi_0\rangle$. We assume always $U \gg W$, such that the difference in potential energy can be neglected in a first step except for the case $\epsilon_\alpha = 0$ which occurs for some special sample geometries and particle numbers. Corrections due to the disorder will be considered in Section 4.2.

In the generic case when $\epsilon_\alpha > 0$ for all intermediate states, the energy differences in the denominator $\mathrm{Den}(S)$ are dominated by the difference in interaction energy. In the limit of strong interaction, one therefore gets

$$
\mathrm{Den}(S) \approx (-U)^{n-1} \prod_{\alpha=1}^{n-1} \epsilon_\alpha . \tag{17}
$$

4 Persistent current

4.1 Longitudinal current

Since the longitudinal persistent current at zero temperature $I = -\partial E_0 / \partial \phi_x$ is given by the flux dependence of the ground state energy, a perturbative treatment of the latter in terms of the hopping t yields a systematic expansion of the persistent current.

Calculating the nth order correction $E_0^{(n)}(\phi_x)$ to the ground state energy, one gets the correction to the persistent current

$$
I^{(n)}(\phi_x) = -\frac{\partial E_0^{(n)}}{\partial \phi_x} \tag{18}
$$

in nth order in the perturbation H_{K}.

4.1.1 Relevant terms of the perturbation theory

When the sequence S of n hopping elements is chosen such that each of the particles returns to its initial position without completing a tour around the ring, every particle which has crossed the boundary must necessarily cross it a second time in the opposite direction such that $h_{\rm b} = h_{\rm f}$. Therefore the flux dependence of these contributions disappears and they cannot influence the persistent current. The lowest order of the perturbation theory which yields a finite contribution $I^{(n)}$ is $n = L_x$, corresponding to the sequences in which one particle starting at (x_0, y_0) crosses the boundary and returns to its original position after completion of its journey around the ring at constant $y = y_0$. If there are more than one particle in the line of the lattice with constant $y = y_0$, a contribution of the same order arises from sequences of hops which move each of the particles to the position of its neighbor (see Fig. 1). Since any hopping in y-direction leads to an increase of the order of the contribution, the lowest order correction to the persistent current is given by the considered processes in which y is kept constant and all of the hops are either in forward or in backward direction.

We now address the dependence of the permutation P_S corresponding to this kind of process on the number N_y of particles in the line at constant y. If $N_y = 1$, the particle returns to its initial site, the final configuration is exactly equal to the initial one and $\mathrm{sign}(P_S) = +1$. In the general case of arbitrary N_y, the considered sequences of hops (see Fig. 1) lead to a cyclic rotation of the order of the N_y particles and the sign of the corresponding permutation is $\mathrm{sign}(P_S) = (-1)^{N_y - 1}$.

This yields the result for the lowest-order ϕ_x-dependence of the ground state energy

$$E_0^{(L_x)}(\phi_x) \approx -\sum_S \frac{t^{L_x}\mathrm{sign}(P_S)\exp\left[-\mathrm{i}\phi_x(h_{\rm f} - h_{\rm b})\right]}{U^{L_x-1}\epsilon_{\alpha_1}\epsilon_{\alpha_2}\ldots\epsilon_{\alpha_3}}. \tag{19}$$

Each sequence considered in this sum contains either L_x forward hops with $h_{\rm f} = 1$; $h_{\rm b} = 0$ or it contains L_x backward hops with $h_{\rm f} = 0$; $h_{\rm b} = 1$. Each given backward sequence $S_{\rm b}$ can now be assigned to the forward sequence $S_{\rm f}$ with the reversed order of hops whose contribution differs only in the sign of the flux-dependent phase-factor. One can express the result as a sum over the forward hopping sequences

$$E_0^{(L_x)}(\phi_x) \approx -2\frac{t^{L_x}}{U^{L_x-1}}\sum_{S_{\rm f}} \frac{\mathrm{sign}(P_S)\cos\phi_x}{\epsilon_{\alpha_1}\epsilon_{\alpha_2}\ldots\epsilon_{\alpha_{L_x-1}}}. \tag{20}$$

Within the above perturbation theory, when processes corresponding to two loops around the ring (which are at least of order $2L_x$) are neglected as compared to the lowest order one-loop processes, the flux dependence of the ground state energy is harmonic and 2π-periodic in ϕ_x.
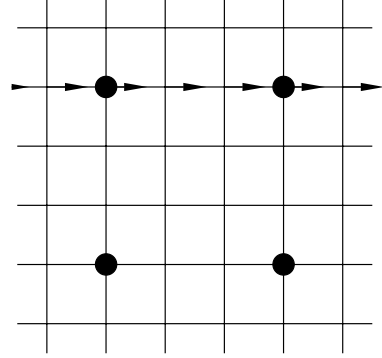

Fig. 1. 4 particles on 6×6 sites. The Wigner crystal configuration has regular square structure. The arrows indicate $L_x = 6$ hops from which a contribution in lowest order to the persistent current arises.

4.1.2 Lowest order result for the persistent current

From (20), one obtains the persistent current in L_xth order perturbation theory

$$I^{(L_x)}(\phi_x) = \tilde{I}^{(L_x)}\sin\phi_x \tag{21}$$

with the flux–independent amplitude

$$\tilde{I}^{(L_x)} \approx -2\frac{t^{L_x}}{U^{L_x-1}}\sum_{S_{\rm f}} \frac{\mathrm{sign}(P_S)}{\epsilon_{\alpha_1}\epsilon_{\alpha_2}\ldots\epsilon_{\alpha_{L_x-1}}}. \tag{22}$$

This result contains several interesting features. First, the absolute value of the persistent current decays proportionally to t^{L_x}/U^{L_x-1}, in the limit of strong interaction. In order to determine the constant prefactor of this power law, it is sufficient to figure out all possible processes which transform the ground state into itself, using L_x forward hopping processes, and to calculate the corresponding ϵ_α from (16).

Furthermore, the sign of the dominating contributions to the persistent current in the limit of strong interaction is given by $-\mathrm{sign}(P_S)$, which for spin-polarized electrons is given by $(-1)^{N_y}$ with the number N_y of electrons in the line of the sample at constant y. This is consistent with the well-known theorem by Leggett for the sign of the persistent current of spinless fermions in 1d [28] (positive $\tilde{I}$ or paramagnetic response for N even and negative or diamagnetic for N odd). Only if the unperturbed ground state (Wigner crystal) configuration of the particles is such that the particle numbers N_y in different occupied lines y have different parity, the prefactors of the corresponding terms have to be considered to determine the sign of the persistent current. For N_y particles in a line of length L_x, the number of hopping sequences $N_{\rm seq}(N_y)$ going from the ground state to itself is the number of terms contributing to the sum in equation (22). Therewith, the result for the persistent current can be roughly estimated to be

$$\tilde{I}^{(L_x)} \propto \frac{t^{L_x}}{U^{L_x-1}}\sum_{y=1}^{L_y} N_{\rm seq}(N_y)(-1)^{N_y}. \tag{23}$$

In the limit of low filling $N_y/L_x \to 0$, $N_{\text{seq}}(N_y)$ is approximatively given by the number of possibilities $N_{\text{seq}}(N_y)$ for N_y particles to each make L_x/N_y (here we assume for simplicity that L_x/N_y is an integer) forward jumps to reach the position of its neighbor, leading to the estimate $N_{\text{seq}}(N_y) \approx L_x!/[(L_x/N_y)!]^{N_y}$. At finite filling, one must consider that the neighbor particle must have left its starting site before the arriving particle can do its last hop. This correlation of the order of the hops of different particles reduces $N_{\text{seq}}(N_y)$. With increasing filling, $N_{\text{seq}}(N_y)$ starts to exponentially increase with N_y and continues to increase more slowly until $N_y = L_x/2$ (half filling). For larger filling it decreases, thereby obeying a symmetry with respect to half filling which is a consequence of the symmetry between particles and holes. One can expect that the contribution of the line $N_y = N_y^{\text{max}}$ with the largest number $N_{\text{seq}}(N_y^{\text{max}})$ of sequences (at low filling this is the one with the maximum number of particles) may dominate over the contributions of the ones with fewer sequences. The sign of the persistent current is then likely to be $(-1)^{N_y^{\text{max}}}$.

4.1.3 Examples

As an example, we calculate explicitly the lowest order term in $\frac{1}{U}$ of the persistent current for 4 spinless fermions on a few small $L_x \times L_y$ rectangular lattices, using the formula (22).

4×2 *sites*
We start with the simple case of a 4×2 lattice. The electrostatically lowest energy configuration (the Wigner crystal) is shown in Figure 2. The number of particles in the two lines at $y = 1$ and $y = 2$ of the system is $N_y = 2$, and the lowest order of the perturbation theory which yields a contribution to the persistent current is $n = L_x = 4$. From these two ingredients, we can immediately determine the sign ($\text{sign}(P_S) = -1$ for N_y even) and the power law of the decrease of the persistent current at strong interaction strength. The leading term of the amplitude (22) is given by

$$\tilde{I}^{(4)} \approx 2\frac{t^4}{U^3} \sum_{S_f} \frac{1}{\epsilon_{\alpha_1}\epsilon_{\alpha_2}\ldots\epsilon_{\alpha_{L_x-1}}}. \qquad (24)$$

This result is always positive and $\propto 1/U^3$. Therefore, the response of the system to the applied flux is always paramagnetic at strong interaction.

For this example, it is an easy exercise to figure out all hopping sequences which contribute in lowest order. For each line $y = 1, 2$, one finds 4 different sequences and can explicitly calculate the interaction energies and ϵ_α of the intermediate states. This allows to evaluate the sum over all sequences in (24) with the result

$$\tilde{I} \to \tilde{I}^{(4)} \approx 853\frac{t^4}{U^3} \qquad \text{for} \qquad \frac{t}{U} \to 0. \qquad (25)$$

In the case of 8 spinless fermions on 4×4 sites, the same analysis can be carried out except for the fact that now
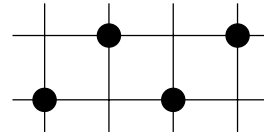

Fig. 2. Ground state configuration for 4 particles on a 4×2 lattice.

four lines $y = 1, 2, 3, 4$ must be considered, yielding an additional factor of 2 in (25). In particular, the persistent current is always paramagnetic in the strong interaction limit, as found numerically in reference [35].

2×4 *sites*
The situation for 4 particles on 2×4 sites is even simpler since there are four lines $y = 1, 2, 3, 4$, each of them containing $N_y = 1$ particle. Since $L_x = 2$, the leading order of the perturbation theory at strong interaction is $n = 2$, and the sequences of hops contain only one intermediate state. The evaluation of (22) yields

$$\tilde{I} \to \tilde{I}^{(2)} \approx -15\frac{t^2}{U} \qquad \text{for} \qquad \frac{t}{U} \to 0, \qquad (26)$$

and the persistent current is diamagnetic at strong interaction.

6×6 *sites*
4 particles on 6×6 sites is the situation investigated numerically in reference [5], at strong disorder. As a function of the interaction strength, an increase of the average persistent current at intermediate strength and a decrease at strong interaction was found. An exponential dependence on the interaction strength was fitted to the data for not too strong interaction. Furthermore, it was noticed that, at strong interaction, the persistent current became paramagnetic for all samples, independent of the disorder realization.

The Wigner crystal ground state on such a lattice is of square form, as shown in Figure 1. Thus, there are two lines with $N_y = 2$ in which hopping sequences of $L_x = 6$ hops can contribute to the persistent current. This explains immediately that the response is paramagnetic, since N_y is even and $-\text{sign}(P_S) = +1$. The persistent current decreases in the strong interaction limit as t^6/U^5. A laborious evaluation of all the 18 sequences for each of the lines with $N_y = 2$ yields the result

$$\tilde{I} \to \tilde{I}^{(6)} \approx 1.808 \times 10^6 \frac{t^6}{U^5} \qquad \text{for} \qquad \frac{t}{U} \to 0. \qquad (27)$$

As compared to the previous cases, the prefactor is much larger. This is caused by the bigger number of contributing sequences and, more importantly, the difference in interaction energy between the intermediate states and the unperturbed ground state ϵ_α. The latter can be very small when the distance between the particles is large.

A numerical investigation by direct diagonalization of the corresponding Hamiltonian matrices [40] confirms the signs, the power laws and the numerical prefactors predicted by the above formulas, also for the last case of 4

particles on 6×6 sites, where the persistent current is indeed found to follow the power law of (27) at strong interaction. However, it must be noticed that the sign of the persistent current is well established [5] at interaction values much lower than the ones where the agreement in amplitude with our formula starts to be good. Even though the data always follow the power laws at strong interaction, fitting an exponential interaction dependence, as done in reference [5], is possible at moderately strong interaction, and might allow to extract useful informations in the regime where higher order terms of the perturbation theory are non-negligible.

4.2 Disorder effects at strong interaction

The role of the disorder and realization-dependent fluctuations of the persistent current vanish in the limit of strong interaction, when $W/U \to 0$. In this section, we treat the lowest order correction in W/U to the results presented above.

In addition, we consider the special case of a perfectly clean system $W = 0$, in which the translational symmetry can considerably influence the interaction dependence of the persistent current.

4.2.1 Disorder corrections to the persistent current

In order to take into account the disorder corrections in the perturbation theory for the longitudinal persistent current, it is not sufficient to consider in the denominator of the expansion terms the ϵ_α which completely neglect the disorder energy. Instead, the full energy differences $E_0 - E_\alpha = -U\tilde{\epsilon}_\alpha$ with

$$\tilde{\epsilon}_\alpha = \frac{1}{U} \left(E_\alpha^{\mathrm{I}} + E_\alpha^{\mathrm{D}} - E_0^{\mathrm{I}} - E_0^{\mathrm{D}} \right) \tag{28}$$

account also for the differences in disorder energy between the intermediate states α and the ground state. The difference

$$\tilde{\epsilon}_\alpha - \epsilon_\alpha = \frac{1}{U} \left(E_\alpha^{\mathrm{D}} - E_0^{\mathrm{D}} \right) = \frac{W}{U} \sum_i v_i \left(n_i(\alpha) - n_i(0) \right), \tag{29}$$

is of the order W/U and vanishes in the limit $W/U \to 0$. With the definition

$$d_\alpha = \sum_i v_i \left(n_i(\alpha) - n_i(0) \right) \tag{30}$$

we can write $\tilde{\epsilon}_\alpha = \epsilon_\alpha + \frac{W}{U} d_\alpha$ and therewith express the energy difference terms in the denominator of (13) as

$$\frac{1}{E_0 - E_\alpha} = \frac{-1}{U\tilde{\epsilon}_\alpha} = \frac{-1}{U\epsilon_\alpha \left(1 + \frac{W}{U\epsilon_\alpha} d_\alpha \right)}. \tag{31}$$

Taking the lowest order term in W/U, and averaging over the ensemble yields

$$\left\langle \frac{1}{E_0 - E\alpha} \right\rangle \approx \frac{1}{-U\epsilon_\alpha} \left(1 - \frac{W}{U\epsilon_\alpha} \langle d_\alpha \rangle \right), \tag{32}$$

where the brackets $\langle \ldots \rangle$ denote the ensemble average over all disorder realizations. At first glance one could expect that the correction linear in W/U vanishes because $\langle d_\alpha \rangle = 0$, when the disorder average is taken over all values of d_α. However, since the ground state is the Wigner crystal pinned at the *lowest* disorder configuration, d_α is more likely positive than negative. The first correction is thus linear in W, and since ϵ_α is always positive, the lowest order correction to the persistent current due to disorder is reducing the contributions of all sequences, the result being

$$I_{\mathrm{W}}^{(L_x)} \approx -2 \frac{t^{L_x}}{U^{L_x-1}} \sum_{S_{\mathrm{f}}} \frac{\mathrm{sign}(P_S)}{\epsilon_{\alpha_1} \epsilon_{\alpha_2} \ldots \epsilon_{\alpha_{L_x-1}}}$$
$$\times \left(1 - \sum_\alpha \frac{1}{\epsilon_\alpha} \frac{W \langle d_\alpha \rangle}{U} + O[(W/U)^2] \right). \tag{33}$$

4.2.2 The particular case $W = 0$

At $W = 0$ we have $H_0 = H_I$, and the ground state at $t = 0$ is degenerate because of the translation symmetry. The number n_{D} of equivalent degenerate Wigner crystal positions and corresponding basis states $|\psi_0^{(\beta)}\rangle$ depends on the system size and the number of particles. The hopping terms however lead to a coupling of these degenerate basis states, and split the degenerate levels, except for special situations where symmetries persist. The coupling terms themselves can be expressed in terms of a perturbative expansion in t. The order p in which the degeneracy of the ground state is lifted may be different from the lowest order n which yields a finite persistent current. We address in the following the three different cases which may occur.

$p > n$ In this case, the splitting can be ignored as compared to the persistent current at strong interaction. The perturbation theory can be applied as in the disordered case, using one of the degenerate ground state configurations (the result will be the same for arbitrary superpositions of the degenerate basis states).

$p = n$ When the persistent current is given by terms which are of the same order as the ones which lift the ground state degeneracy, the flux dependent correction to the ground state energy and the persistent current are given by the lowest eigenvalue of the effective coupling matrix M between the n_{D} degenerate basis states. The matrix elements are

$$M_{\beta,\beta'} = \sum_{\alpha_1,\alpha_2,\ldots,\alpha_{p-1}} \tag{34}$$
$$\frac{\langle \psi_0^{(\beta)}|H_{\mathrm{K}}|\psi_{\alpha_1}\rangle \langle \psi_{\alpha_1}|H_{\mathrm{K}}|\psi_{\alpha_2}\rangle \ldots \langle \psi_{\alpha_{p-1}}|H_{\mathrm{K}}|\psi_0^{(\beta')}\rangle}{(E_0 - E_{\alpha_1})(E_0 - E_{\alpha_2}) \ldots (E_0 - E_{\alpha_{p-1}})}.$$

Since in this case all matrix elements are $\propto t^n/U^{n-1}$, the lowest eigenvalue and thus the persistent current will follow such a power law too.

$p < n$ When the levels are split at an order which is lower than the one which yields contributions to the persistent

current, the higher order terms must be calculated using the lowest energy ground state found from the diagonalization of M. Such a ground state will be a superposition

$$|\psi_0\rangle = \sum_{\beta=1}^{n_{\mathrm{D}}} f_\beta |\psi_0^{(\beta)}\rangle \qquad \text{with} \qquad \sum_\beta^{n_{\mathrm{D}}} |f_\beta|^2 = 1 \qquad (35)$$

of the different Wigner crystal positions. Plugging this ground state into the general expression for the corrections to the ground state energy (13), one gets

$$E_0^{(n)}(W=0) = \sum_{\beta,\beta'} f_\beta^* f_{\beta'} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_{n-1}} \qquad (36)$$

$$\frac{\langle\psi_0^{(\beta)}|H_{\mathrm{K}}|\psi_{\alpha_1}\rangle\langle\psi_{\alpha_1}|H_{\mathrm{K}}|\psi_{\alpha_2}\rangle\ldots\langle\psi_{\alpha_{n-1}}|H_{\mathrm{K}}|\psi_0^{(\beta')}\rangle}{(E_0 - E_{\alpha_1})(E_0 - E_{\alpha_2})\ldots(E_0 - E_{\alpha_{n-1}})},$$

and realizes that contributions can arise from hopping processes starting at any of the ground state components $|\psi_0^{(\beta)}\rangle$, and ending at an arbitrary $|\psi_0^{(\beta')}\rangle$ where β and β' can be equal or different. The processes with $\beta \neq \beta'$ can in some special cases have a lower power in $\frac{1}{U}$ than the processes with $\beta = \beta'$, relevant in the disordered case, and therefore dominate in the limit of strong interaction.

The case $W = 0$ can be qualitatively different from the disordered case $W \neq 0$, even at very weak disorder. The reason for this is the fact that the unperturbed ground state is a superposition of the degenerate positions of the Wigner crystal. This degeneracy is lifted by the disorder, and the typical energy difference between two positions is of the order $W\sqrt{N}$ (this arises from the N different random on-site energies, all being of order W). The coupling between these electro-statically equivalent states is provided by the hopping terms and can itself be estimated from a perturbation theory. The lowest order coupling needs a number of hops which is typically $p = N$ since the Wigner crystal must in most cases be translated as a whole and each of the particles must hop at least once. In some cases, a $p < N$ can be sufficient, as in the example of 3 particles on 4×2 sites (see below).

Then, the coupling decreases at strong interaction as t^p/U^{p-1} and becomes, in the limit of strong interaction U, always much smaller than the splitting due to the disorder. This holds at any finite disorder value (provided $p > 1$), and prevents a mixing of the different Wigner crystal configurations in the ground state. Only for $W = 0$, the splitting by the disorder vanishes, and an infinitesimal coupling leads to the superposition of the different configurations.

4.2.3 Example for disorder corrections

4×2 *sites, 4 particles*
The disorder dependence of the flux sensitivity is illustrated for the example of 4 particles in $(L_x, L_y) = (4, 2)$. One expects an $1/U^3$ dependence for the leading term of the persistent current (24) and, according to (33), a W/U^4 dependence for the linear correction in disorder.
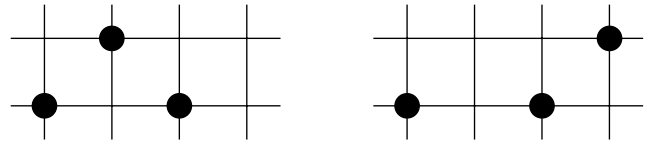

Fig. 3. Two degenerate configurations for 3 particles on 4×2 sites which can be linked to each other by sequences of only two hopping processes.

We have performed numerical calculations at strong interaction [40], which are in good agreement with these power laws predicted by the lowest order of the perturbation theory. The persistent current decreases when the strength of the disorder is increased, as predicted by the negative correction in formula (33).

Effects of degeneracies at $W = 0$
The clean case however has a very different amplitude due to the degeneracies in the Wigner crystal. The persistent current can be calculated from (36), and after considering all the possible sequences between the different degenerate configurations, one obtains $\tilde{I}^{(4)}(W = 0) \approx 2.1238\,\tilde{I}^{(4)}$, with $\tilde{I}^{(4)}$ taken from (24). A numerical check [40] confirms this result at strong interaction.

4×2 *sites, 3 particles, $W = 0$*
In order to show that the absence of disorder does not only change the prefactor of the interaction dependence at large U, but can even influence the power of the U-dependence, we consider 3 particles in $(L_x, L_y) = (4, 2)$. The particle which is alone in one line has two possible positions which are degenerate at $t = W = 0$, and there exists a sequence of only two one-particle hops which links the two degenerate states with each other (see Fig. 3). Since the ground state at $W = 0$ is a superposition of all possible degenerate configurations, this process leads to a contribution $\propto t^2/U$ to the persistent current which dominates at strong interaction. In the disordered case, in contrast, the degeneracy of the ground state configurations is lifted and only sequences consisting of at least four hops can link the ground state to itself. Then, the interaction dependence of the lowest order contribution to the persistent current is $\propto t^4/(WU^2)$. Interestingly, in this example, one of the intermediate states touched by a sequence of 4 hops can be a different, but electro-statically equivalent configuration of the Wigner crystal, and $\epsilon_\alpha = 0$, such that even at $U \gg W$, the disorder energy cannot be neglected.

A numerical check [40] shows that by adding the disorder, the asymptotic dependence indeed follows this prediction and goes from $\propto U^{-1}$ to $\propto U^{-2}$.

4.3 Transverse current

4.3.1 General considerations

Without disorder, the current in y-direction must vanish because of the symmetry of the problem (no flux is applied in y-direction). While this remains true in the ensemble
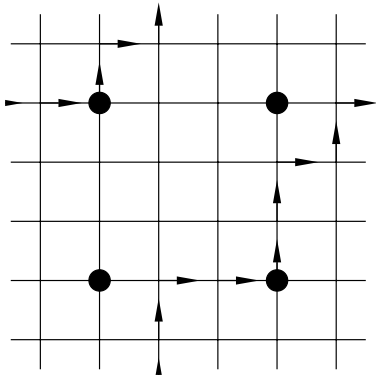
Fig. 4. A sequence of hopping processes which gives rise to a finite contribution to the transverse current in the disordered system of 4 particles on 6×6 sites. The arrows indicate $L_x + L_y = 12$ forward and upward hops.

average, the presence of disorder breaks this symmetry in individual samples. A finite transverse current can be observed in addition to the longitudinal current driven by the flux ϕ_x. The transverse current can be calculated from

$$I_t(\phi_x) = -\left.\frac{\partial E_0(\phi_x, \phi_y)}{\partial \phi_y}\right|_{\phi_y=0}, \qquad (37)$$

such that a theoretical approach to determine $I_t(\phi_x)$ consists in using the perturbative expansion for the ground state energy, as for the longitudinal current, but with a finite ϕ_y. Then, the derivative with respect to ϕ_y at $\phi_y = 0$ can be evaluated for the leading contribution in the strong interaction regime. Therefore, only terms in the series (13) which depend on ϕ_y can make a finite contribution to the transverse current.

In order to get such contributions, one needs hopping sequences in y-direction which contain hops crossing the boundary between sites (x, L_y) and $(x, 1)$ with $h_\mathrm{u} - h_\mathrm{d} \neq 0$. h_u and h_d denote the number of such hops from $y = L_y$ to $y = 1$ ('upwards') and from $y = 1$ to $y = L_y$ ('downwards').

However, the flux dependence of the energy due to sequences containing L_y upward or L_y downward hoppings is $\propto \cos\phi_y$, as for the longitudinal processes (20), which are $\propto \cos\phi_x$. As a consequence, the resulting transverse current from these sequences is $\propto \sin\phi_y$, and vanishes when $\phi_y = 0$ is taken, just like the longitudinal current (21) at $\phi_x = 0$. Therefore, no transverse current can be obtained in the order L_y of the perturbation theory.

Nevertheless, higher order sequences exist which do lead to finite contributions. These are due to sequences of hops which cross both, the boundary in x, and the boundary in y-direction. The lowest order of the expansion in which transverse currents appear is $m = L_x + L_y$, corresponding to sequences which consist of a combination of only forward and downward hops, only backward and upward hops, or vice versa, which complete one round in both spatial dimensions of the system. An example of such a sequence is shown in Figure 4.

As before, each of the sequences containing only forward and upward hops S_fu can be associated with the reverse process consisting of backward and downward hops S_bd. The contribution of the reverse process differs only in the sign of the flux-dependent phase factors. The correction to the ground state energy (13) of all S_fu and S_bd sequences together can be written as

$$E_{0,\mathrm{fu+bd}}^{(m)} = -2\frac{t^m}{U^{m-1}}\cos(\phi_x + \phi_y)\sum_{S_\mathrm{fu}}\frac{\mathrm{sign}(P_{S_\mathrm{fu}})}{\tilde{\epsilon}_{\alpha_1}\dots\tilde{\epsilon}_{\alpha_{m-1}}}. \quad (38)$$

The same association can be made for sequences containing only forward and downward hops S_fd, which are regrouped with the corresponding reverse processes consisting of backward and upward hops, yielding

$$E_{0,\mathrm{fd+bu}}^{(m)} = -2\frac{t^m}{U^{m-1}}\cos(\phi_x - \phi_y)\sum_{S_\mathrm{fd}}\frac{\mathrm{sign}(P_{S_\mathrm{fd}})}{\tilde{\epsilon}_{\alpha_1}\dots\tilde{\epsilon}_{\alpha_{m-1}}}. \quad (39)$$

4.3.2 Lowest order result for the transverse current

By taking the derivative with respect to ϕ_y at $\phi_y = 0$, we obtain for the transverse current in mth order of the perturbation theory

$$I_t^{(m)}(\phi_x) = 2\frac{t^m}{U^{m-1}}\sin\phi_x \qquad (40)$$

$$\times \left(\sum_{S_\mathrm{fu}}\frac{\mathrm{sign}(P_{S_\mathrm{fu}})}{\tilde{\epsilon}_{\alpha_1}\dots\tilde{\epsilon}_{\alpha_{m-1}}} - \sum_{S_\mathrm{fd}}\frac{\mathrm{sign}(P_{S_\mathrm{fd}})}{\tilde{\epsilon}_{\alpha_1}\dots\tilde{\epsilon}_{\alpha_{m-1}}}\right),$$

with $m = L_x + L_y$ being the lowest order which yields a non-zero contribution. Even when $U \gg W$, the disorder energy cannot be neglected as in the lowest order term for the longitudinal current, since the two contributions to the current in (40) cancel each other exactly when the disorder vanishes. It can also be seen that the lowest order result for the transverse current is proportional to $\sin\phi_x$ and vanishes with the longitudinal current at $\phi_x = 0$.

4.3.3 Example

In the case of 4 particles in a lattice of 6×6 sites, we have seen above (see Eq. (27)) that the longitudinal persistent current decreases at strong interaction following the power law $\propto t^6/U^5$. The suppression of the transverse current is much more pronounced, since the leading contribution at strong interaction is of the order $L_x + L_y = 12$, and our theory (40) predicts that it decays $\propto t^{12}/U^{11}$. This means that the particle mobility is more and more restricted to the longitudinal direction when the interaction is increased. That the suppression of the transverse current is much stronger than the one of the longitudinal persistent current has been noticed in the numerical study of reference [5]. The dominance of the longitudinal current at strong interaction also explains the observation of "plastic flow" in a study of the local persistent currents [35]. The orientation of the local currents has been proposed as a signature of the insulator-metal transition occurring in interacting 2d systems [6].

4.4 Persistent current for electrons with spin

In this section we present some examples of the effects of the spin of the electrons on the longitudinal persistent current, showing that our approach is not restricted to spinless fermions, but can also be used for electrons carrying spin.

For finite systems containing N particles, it is well known that the spin polarization of the ground state can depend on the magnetic flux due to level crossings between states of different spin symmetry as a function of the flux [33]. However, without an external magnetic field acting on the electrons in the ring, there is a degeneracy related to the operator S_z in the subspace of fixed total spin and one can write $E(S^2, S_z) = E(S^2)$.

It is in principle possible (though difficult experimentally) to create a magnetic flux through the ring while the magnetic field remains vanishing at the positions of the electrons in the ring. Such a flux does not lift the spin degeneracy mentioned above. If we now want to follow the dependence of the ground state energy on the flux, we can restrict the study to the subspace of minimum absolute value of S_z, choosing $S_z = 0$ for an even number of particles with $N_\uparrow = N/2$ spins up and $N_\downarrow = N/2$ spins down, or $S_z = 1/2$ for an odd number of particles with $N_\uparrow = \frac{N+1}{2}$ spins up and $N_\downarrow = \frac{N-1}{2}$ spins down.

4.4.1 One-dimensional systems

In 1d, a general rule exists for the sign of the persistent current in the case of spinless fermions [28]. This rule is valid at arbitrary disorder and interaction. Below, we address the question of the sign of the persistent current for electrons with spin in different parameter regimes.

Non-interacting electrons with spin
In the case of non-interacting electrons with spin, one can separately consider the flux dependence of the energy of the particles with up spin and the one of the particles with down spin, and add up their contributions to the persistent current.

If N is even, the contribution of the $N/2$ up-spins to the persistent current will have the same sign $(-1)^{N/2}$ as the contribution of the $N/2$ down-spins. This determines the sign of the persistent current. For odd N, the contribution of the $(N + 1)/2$ up spins will have the sign $(-1)^{(N+1)/2}$ while the down spins contribute a term with sign $(-1)^{(N-1)/2}$. In this case, to know the sign of the persistent current one must compare the amplitudes of the two contributions and the result will in general depend on the disorder configuration. However, in the zero-disorder case at low filling, the sign of the persistent current around flux $\phi = 0$ is known to be paramagnetic for odd $N > 1$ [30].

N even, strong interaction
At strong interaction, the charges form a Wigner crystal which is pinned by the disorder. In this limit, the spin degree of freedom can be treated by an effective spin Hamiltonian. The latter turns out to be the anti-ferromagnetic Heisenberg Hamiltonian when $\phi = 0$, the positions $i = i_1, i_2, \ldots, i_N$ of the charges of the Wigner crystal being the spin lattice sites. Thus, the expectation values of the occupation number operators vanish everywhere except on these sites where they satisfy $\langle \hat{n}_{i_k,\uparrow} + \hat{n}_{i_k,\downarrow} \rangle = 1$. For an even number N of spins in 1d, according to a theorem by Marshall, the ground state $|\psi_0\rangle$ of this spin Hamiltonian is a singlet of total spin $S = 0$ [41], and can be expressed in the form

$$|\psi_0\rangle = \sum_\beta f_\beta \prod_{i=i_1}^{i_N} \left(c_{i,\uparrow}^+\right)^{n_{i,\uparrow}(\beta)} \prod_{i=i_1}^{i_N} \left(c_{i,\downarrow}^+\right)^{n_{i,\downarrow}(\beta)} |0\rangle \quad (41)$$

with real $f_\beta > 0$ for all spin configurations β with fixed $S_z = 0$. Note that in this expression, the ordering of the operators is done firstly according to spin, and secondly according to the position.

As in the case of spinless fermions without disorder, the ground state is a superposition of different basis states and the lowest order contributions to the ground state energy, which are flux dependent, are again of the order $n = L_x$. Similar to (36), they can arise from different kinds of sequences. First, only the up spins are moved around the ring, giving rise to a cyclic permutation of the spin up operators in (41), yielding the sign $(-1)^{N_\uparrow}$. The sequences involving only down spins give the same result. In addition, a sequence which moves all of the particles to the position of their neighbor can also contribute to the flux dependence since the resulting spin configuration is also contained in the ground state. The sign however is the same as the one of the previous sequences, since only one of the particles crosses the boundary and therefore only the order of the operators for one spin direction has to be restored by the corresponding cyclic permutation, and because the prefactor $f_\beta^* f_{\beta'}$, arising from the weights of the different components in the ground state, is always real and positive.

Since all contributions have the same sign, the sign of the persistent current in one-dimensional disordered chains at strong interaction is always given by this sign $(-1)^{N/2}$, provided the particle number is even, independent of the particle density N/L. Because the ground state (41) holds only at $\phi = 0$, this sign rule is granted only in the vicinity of $\phi = 0$.

The resulting sign is the same as the one found in the non-interacting case [30], and for the Hubbard model at half filling [31], but it differs from the result for the Hubbard model at low filling [32,42], where the current around $\phi = 0$ is found to be diamagnetic for even numbers of particles at strong interaction. This difference may be due to the fact that the Hubbard interaction is local, making the Hubbard model at low filling in the $U \to \infty$ limit equivalent to spinless fermions. The more realistic long-range Coulomb interactions considered here do not show this artifact and lead to a qualitatively different behavior. The result is also different from the one obtained for Coulomb interacting electrons with spin in a continuous model with one barrier [33], which is always found to be diamagnetic at strong interaction. In this case, one may attribute the result to the fact that the interaction could lead to a rigid

Wigner crystal which, in a continuous model, would be equivalent to one single heavy particle.

4.4.2 Two-dimensional systems

For two dimensional systems, there is the same degeneracy in the position of the up and the down spins in the minimum S_z subspace. In the strong interaction limit, each line of the Wigner crystal should follow the one dimensional law explained above, but the different spin configurations contributing to the ground state in 2d may have different numbers of spin up and spin down electrons in a given line. This yields contributions of different signs to the persistent current which have to be considered explicitly. It seems not to be possible to provide a simple sign rule. However, the power law decay of the persistent current at large interaction strength should have the same exponent as in the spinless case.

4.5 Size dependence and limitation of the theory

Here, we address the size dependence of the persistent current and the limitation of our theory which is given by the presence of defects in the Wigner crystal.

4.5.1 Localization length for the Wigner crystal

From the exponential size-dependence of equation (22), one can extract the localization length for the Wigner crystal. A comparison to the exponential size scaling

$$I(L_x) \propto \exp(-L_x/\xi) \qquad (42)$$

yields for large L_x a localization length

$$\xi = \left(\ln \left(\frac{U}{t} \right) \right)^{-1}, \qquad (43)$$

showing the same decrease with increasing interaction strength as the localization length of the Mott insulator appearing in the Hubbard model at strong interaction [23].

4.5.2 Defects in the Wigner crystal

In the thermodynamic limit, the Wigner crystal is no longer a single domain and the above theory cannot be applied in the present form. At any finite, even very strong interaction, defects and domain walls can arise which allow to gain an amount of potential energy which grows with the size of the domains. The crystal might then prefer to be divided in defect-free domains each being pinned by the disorder. In the weak disorder limit, one can use similar arguments as in [43] in order to estimate the size of such domains. In the following, we briefly address different types of defects and calculate the associated critical ratio $\left(\frac{U}{W} \right)_{\rm c}$ above which the theory applies.

Point defects
In the case of point-like defects, consisting of a single charge of the Wigner crystal being displaced by one lattice constant a with respect to its position in the perfect crystal, the gain in disorder potential energy can be estimated to be $\Delta E_{\rm disorder}^{\rm point} \approx W$. On the other hand, the cost in interaction energy is $\Delta E_{\rm interaction}^{\rm point} \approx \frac{Ua^2}{b^3}$ with b being the mean spacing between particles. Since the charge density is given by $\nu = 1/b^2$, this yields the density depending criterion

$$\frac{U}{W} > \left(\frac{U}{W} \right)_{\rm c}^{\rm point} \approx \frac{1}{a^2 \nu^{3/2}} \qquad (44)$$

for the stability of the Wigner crystal against the creation of point defects.

Domain walls
The creation of a domain wall costs an interaction energy which is of the order $\Delta E_{\rm interaction}^{\rm wall} \approx \frac{Ua^2}{b^3} L^{\rm wall}$ with $L^{\rm wall}$ being the length of the wall. By deplacing a domain of linear size R containing $N_d = R^2 \nu$ particles, one can typically gain an amount $W \sqrt{N_d}$ of disorder potential energy. However, the first domain-like defect will appear in the most favorable position. Since the maximally possible gain is the extremely improbable value $W N_d$, one can speculate that the gain in the optimal position is given by an intermediate power $\Delta E_{\rm disorder}^{\rm domain} \approx W N_d^\gamma$ with $1/2 \leq \gamma < 1$. The length of the corresponding wall around the domain is $L^{\rm wall} \sim \sqrt{N_d}$ such that this results in the stability of the crystal against the creation of domains of size R for

$$\frac{U}{W} > \left(\frac{U}{W} \right)_{\rm c}^{\rm domain} (R) \approx \frac{R^{2\gamma-1}}{a^2 \nu^{2-\gamma}}. \qquad (45)$$

At a fixed W, if $U > U_{\rm c}^{\rm domain}(L)$, a crystal of size L will not be perturbed by domains. Otherwise, one can use $U = U_{\rm c}^{\rm domain}(R_{\rm c})$ to extract the typical domain size $R_{\rm c}$.

In systems larger than $R_{\rm c}$, the electron crystal is divided in many domains of size $R_{\rm c}$. If we neglect the coupling energy between these domains, a rough estimate for the response to a flux ϕ_x in x-direction of a system of size (L_x, L_y) is to take the product of the amplitudes of the longitudinally aligned $L_x/R_{\rm c}$ domains and to sum over the responses of the $L_y/R_{\rm c}$ "channels".

Although we have not addressed all possible kinds of defects, we can assume that there is always a critical threshold $\left(\frac{U}{W} \right)_{\rm c}^{\rm defect}$ above which our perturbative theory applies.

5 Summary

We have presented a systematic perturbative treatment of persistent currents in 2d lattice models for the case of strong Coulomb interaction, when the electronic charge density forms a Wigner crystal. The contribution with the weakest interaction dependence corresponds to sequences of one-particle hops along the shortest paths around the ring. These sequences dominate in the limit of strong interaction and determine the sign of the persistent current. For spinless fermions, this sign follows simple rules which depend only on the structure of the Wigner crystal.

Furthermore, we have shown that, except special cases, the leading term for the persistent current and therewith the sign of the persistent current at strong interaction do not depend on the realization of the disordered potential. Only the complete absence of disorder can qualitatively change the behavior. We considered the disorder corrections systematically and showed that they decay as W/U at strong interaction.

In addition, we have shown that transverse currents appearing in individual disordered samples are suppressed much faster than the longitudinal current, thereby establishing an orientation of the local currents in longitudinal direction when the interaction is increased. This explains the numerical observation of the realization-independent sign of the persistent current and of the orientation of the local currents in longitudinal direction reported in references [5, 6, 34, 35].

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